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CENTRIFUGAL AND THERMAL STRESSES

IN CONICAL TURBINE DISKS

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ABSTRACT

Charts are presented for the rapid determination of the stress distributions in solid conical disks. By means of dimensionless coefficients the centrifugal stresses are obtained for imposed radial stresses at the outer periphery of a rotating disk. The dimensionless coefficients for the thermal stresses hold for temperature distributions that are power functions of the disk radius. It is shown that such temperature distributions closely approximate the actual conditions in cooled turbine disks. The method is explained by means of an example which shows the difficulties that are associated with the interpretation of the results by the design engineer.

Relations are presented also for the determination of the centrifugal and thermal stresses in conical disks that have a central bore.

NOMENCLATURE

a_i	coefficients
b	width of rim of turbine disk (in.)
h	blade height
h_c	film coefficient of cooling on face of disk (Btu/(hr, sq.ft, ^oF))
h_o	film coefficient of heating on outer periphery of disk (Btu/(hr, sq.ft, ^oF))
k	coefficient of thermal conductivity (Btu/(hr, ft, ^oF))
$k_{1,2}$	constants
n	exponent of assumed temperature distribution law
q	heat flux (Btu/hr)
$q_{1,2,3}$	coefficients of Table I
r	radius (in.) or (ft.)
r_h	hub radius of blading (in.)
r_t	tip radius of blading (in.)
$s_{1,2,3}$	coefficients of Table I
$t=r/R$	dimensionless radius ratio of conical disk
u	radial displacement at radius r
x	length (Eq. 24) (ft.)
y	thickness of disk (in.) or (ft.)
A	blade cross section area (in.^{2})
A_h	blade cross section area at hub (in.^{2})
A_t	blade cross section area at tip (in.^{2})
$A_{1,2}$	constants
$B_{1,2}$	constants
C	constant

E	modulus of elasticity (psi)
F_B	centrifugal force of blade (lb)
F_R	centrifugal force of rim of turbine disk (lb)
$H_{C\ 1,2}$	dimensionless coefficient for centrifugal stress in tangential direction
H_T	dimensionless coefficient for thermal stress in tangential direction
$I_{o,1}$	modified Bessel functions of zero and first order
K	constant
R	radius of apex of conical disk (in.)
$R_{C\ 1,2}$	dimensionless coefficient for centrifugal stress in radial direction
R_T	dimensionless coefficient for thermal stress in radial direction
T	temperature (^oF)
T_c	coolant temperature (^oF)
T_g	gas temperature (^oF)
U	peripheral speed of disk (in./sec.) or (ft./sec.)
α	coefficient of thermal expansion (%/^oF)
ϵ	strain
ζ	ratio of sum of areas A_h and outside rim area of disk
$\eta_{1,2}$	coefficients
θ	temperature difference
$\varkappa = A_t/A_h$	ratio of blade cross section areas at tip and hub
ν	Poisson's ratio
ρ	mass density of disk material (lb. sec^2/in.^{4})
ρ_B	mass density of blade material (lb. sec^2/in.^{4})
σ_B	centrifugal stress of blade at hub (psi)
σ_e	equivalent stress (psi)

σ_o imposed radial stress at outer radius of disk (psi)

σ_r radial disk stress (psi)

σ_t tangential disk stress

$\sigma_u = \rho U_o^2$ reference stress (psi)

ω angular velocity of rotation of disk (rad./sec.)

Subscript

av refers to : mean radius of blading

i : inner radius of disk

m : center line of disk

o : outer radius of disk

C : centrifugal

T : thermal

INTRODUCTION

Honegger [1][1] solved the general differential equation for the centrifugal stresses in conical disks by hypergeometric series. His results were used by Keller [2], [3], and Salzmann [4] for the determination of the centrifugal stresses in arbitrary disks which are subdivided into a small number of conical sections. With hand calculations this method is easier to handle than that of Grammel [5] which replaces a given disk by a large number of rings with constant thickness. Manson [6] uses Grammel's approach to obtain the centrifugal and thermal stresses in disks with an arbitrary shape. Manson's calculating procedure can be used for any given temperature distribution in radial direction, and for changes of the physical properties of the disk material with temperature.

Although Manson's procedure can be handled with ease by a digital computer, it would often be desirable to have available a simpler method for the preliminary dimensioning of turbine disks. Turbines for space power plants, in particular, must run at the highest possible peripheral speeds and temperatures, to reduce the number of stages and to arrive at a reasonable size of the radiative heat sink. High peripheral speeds are made possible if the disks are cooled. It must then be investigated, however, whether excessive cooling may not produce a situation where the increase in permissible stress due to the lower disk temperatures is offset by the thermal stresses.

Fig. 1 shows a typical design of a high-performance turbine wheel. The blades are attached by fir-tree roots. The disk has no central

[1] Numbers in brackets designate References at end of paper.

hole, and is connected to the shaft by tie-bolts and radial keys, so that concentricity is maintained at all operating conditions. The stress concentrations because of the bolt holes are not dangerous for disk materials with high ductility.

Since tangential stresses cannot occur outside of the radius r_o, the turbine disk of Fig. 1 can be approximated by a conical disk with axial widths y_o and y_i, that is subjected to a uniform radial stress σ_o at the outer radius r_o. The stress σ_o is produced by the centrifugal forces of the blades and the rim of the disk outside of r_o. Disks of this type, with and without a central hole of radius r_i, will be treated in this paper (see Fig. 2).

The mathematical analysis of the centrifugal and the thermal stresses in such disks is given in Appendix I. The derivations show that the total stresses are obtained by adding the centrifugal stresses in a disk of uniform temperature to the thermal stresses of the stationary disk.

CENTRIFUGAL STRESSES

Eqs. 13 give the radial and tangential stresses $(\sigma_r)_C$ and $(\sigma_t)_C$ of the disk of Fig. 2. The reference radius R is

$$R = r_o \frac{1 - (y_o/y_i)\ (r_i/r_o)}{1 - (y_o/y_i)}$$

The coefficients q and s of Eqs. 13 for Poisson's ratio $\nu = 0.3$ are listed in Table I as functions of $t = r/R$, where r is an arbitrary radius between r_i and r_o. The quantities A_1 and A_2 in Eqs. 13 are constants that depend on the stress conditions at r_i and r_o. At $t_o = r_o/R$ the radial stress $(\sigma_r)_C$ equals σ_o, and at $t_i = r_i/R$ the radial stress is usually zero. For known constants A_1 and A_2 the stresses at an arbi-

trary radius r are obtained from Eqs. 13 with the values of q and s from Table I for $t = r/R$.

For solid conical disks with $r_i = 0$ the centrifugal stresses are given by Eqs. 16. These stresses are referred to the hoop stress $\sigma_u = \rho\,\omega^2\,r_o^{\,2} = \rho U_o^{\,2}$ of a thin rotating ring of radius r_o that has the same mass density ρ as the disk. The coefficients R_C and H_C of Eqs. 16 are therefore dimensionless quantities. They are functions of y_o/y_i and r/r_o only, and are plotted in Figs. 3 and 4 for $\nu = 0.3$. If the imposed radial stress σ_o is zero, the radial and tangential disk stresses are $(\sigma_r)_C = \sigma_u\,R_{C1}$ and $(\sigma_t)_C = \sigma_u\,H_{C1}$, respectively. Hence, the coefficients R_{C2} and H_{C2} represent the contributions of σ_o to the disk stresses. It is of interest to note that in a disk with constant thickness ($y_o/y_i = 1$) the coefficients R_{C2} and H_{C2} are unity. Hence, in such disks the stress σ_o must be added to the radial and tangential stresses for $\sigma_o = o$ at all radii.

For a particular turbine wheel the ratio σ_o/σ_u is independent of the rotational speed of the disk. It depends only on the blade and root dimensions, and on the ratio of the mass densities of the blade and disk materials. Although the stress distribtuion in a conical disk with a given thickness ratio y_o/y_i is independent of the actual width y_o, the stress σ_o depends on y_o. Thus, if an initial estimate of σ_o/σ_u gives excessive stresses it is possible, in most cases, to increase simply the width y_o to obtain permissible stress levels.

THERMAL STRESSES

Appendix I shows that it is possible to obtain solutions for the thermal stresses in conical disks for a temperature distribution of the

type $T = T_m + \Delta T_o (r/r_o)^n$, where the exponent n is any whole number. This relation is plotted in Fig.5. It will be shown below that these temperature distributions are very similar to those which can be expected in cooled turbine disks. As shown in Fig. 6, the temperature T_m is the extrapolated disk temperature at $r = 0$, and ΔT_o is the temperature difference between the center line and the outer radius r_o of the disk. Eqs. 21 give the radial and tangential thermal stresses $(\sigma_r)_T$ and $(\sigma_t)_T$, respectively, in conical disks with central holes. The constants B_1 and B_2 are obtained from the condition that the thermal stresses $(\sigma_r)_T$ in radial direction are zero at the inner and the outer radius r_i and r_o . If B_1 and B_2 are known, the distribution of the thermal stresses along the disk radius can be determined with the values of q and s of Table I.

For solid conical disks the thermal stresses are obtained directly from Eqs. 22. They are referred to the stress $\alpha \Delta T_o E$ which occurs in a restrained bar, having the same thermal coefficient of expansion α and modulus of elasticity E as the disk material, which is heated up by the temperature ΔT_o. The highest thermal stresses in the disk cannot exceed this reference stress. The dimensionless coefficients R_T and H_T of Eqs. 22 are functions of y_o/y_i and r/r_o only. Figs. 7 to 10 show these coefficient for $n = 1$ to $n = 4$ for Poisson's ratio $\nu = 0.3$.

Simple relations are obtained for the thermal stresses in disks with constant thickness (see Eqs. 23). The corresponding coefficients R_r and H_T are plotted in Fig. 11 for a range of n from unity to infinity. Because of the relatively small influence of the thickness ratio y_o/y_i on the thermal stresses it is possible to use Figs. 7 to 10 in conjunction with Fig. 11 for the determination of R_T and H_T for arbitrary values of n by means of interpolations.

TEMPERATURE DISTRIBUTION IN COOLED TURBINE DISKS

Appendix II gives a simplified analysis of the temperature distribution in turbine disks with constant thickness, that are cooled in one face, in the manner illustrated by Fig. 12a. Heat from the hot gases enters the disk at its outer periphery through the blades. This heat flows radially inward and is transmitted to the coolant. The film heat transfer coefficient from the hot gases to the blades, and that from the face of the disk to the coolant are taken as constant values. For constant gas and coolant temperatures the temperature distribution in the disk is given by Eq. 28. This relation is used to determine the temperatures in the disk of Fig. 12a, for gas and coolant temperatures of $1500^{o}F$ and $500^{o}F$, respectively. The heat transmitted from the gas to the blades, and thence to the disk, is expressed by an equivalent film coefficient h_o between the gas and the outer periphery of the disk. Reasonable values for h_o , and for the film coefficient h_c between disk face and coolant, are 350 and 100 Btu/(hr, sq. ft, ^{o}F) respectively, in gas turbines.

Disk materials for high temperature applications are usually alloys with a Nickel or Chromium base. All these materials have coefficients of thermal conductivity of about $k = 12$ Btu/(hr, ft, ^{o}F) [10] . An exception is the so-called DuPont TD Nickel with a value of k of about 25 at $1100^{o}F$. Interesting disk materials for extreme temperatures are Molybdenum and its alloys. Their use is restricted, however, to non-oxidizing fluids for temperatures above $1000^{o}F$. Besides their higher strength these materials have coefficients of thermal conductivity of about 50 Btu/(hr, ft, ^{o}F). The curves of Fig. 13 show the temperature distributions of Eq. 28 for $k=12$ and

$k = 50$. Not only are the maximum temperatures lower for the higher value of k, but the temperature gradients are greatly reduced. The circles and dots in Fig. 13 represent the relation $T = T_i + \Delta T_o (r/r_o)^n$. In particular, the dots correspond to $n = 4$, and the circles to $n = 2.5$, where the values of ΔT_o are 710°F and 405°F, respectively. Fig. 13 shows that the temperature distributions in cooled turbine disks can be approximated quite closely by the power relation of Eq. 17 for which the thermal stresses, and the dimensionless coefficients R_T and H_T of Figs. 7 to 11, are presented for disks without central holes.

EXAMPLE

The stresses will be determined for the disk of Fig. 1 which is supposed to have the following dimensions:

r_o = 5 in.	b = 0.6 in.
r_h = 5.6 in.	y_o = 0.4 in.
r_t = 6.6 in.	y_i = 0.8 in.

The mean radius r_{av} of the blading is 6.1 in., its height h is 1 in.. The stress conditions are to be found for peripheral speeds of 800, 1000, and 1200 ft/sec at the mean radius r_{av}, for the temperature distribution of Fig. 13 for $k = 12$, or $T = 537 + 710 (r/r_o)^4$.

For tapered blades with a linearly changing cross section area A from the hub to the tip the centrifugal blade stress σ_B at the radius r_h is

$$\sigma_B = \rho_B \frac{\omega^2}{2} \left(r_t^2 - r_h^2\right) \left(1 - \varkappa \frac{2 + 3 r_h/h}{3 + 6 r_h/h}\right)$$

where ρ_B is the mass density of the blade material, and $\varkappa$ the value $(1 - A_t/A_h)$ of the blade cross section areas at the tip and the hub dia-

meters. The sum of the cross section areas A_h of all the blades of the wheel can be expressed by the fraction ζ of the outer rim area $2 \pi r_h b$. Approximate values of ζ are 0.45 for impulse bladings, and 0.30 for 50% reaction bladings. The total centrifugal force F_B of the blades is then

$$F_B = \zeta \sigma_B 2 \pi r_h b$$

The centrifugal force F_R of the rim between the radii r_o and r_h is roughly

$$F_R = \left(\frac{\rho + \rho_B}{2} \right) \omega^2 \; 2 \pi b \left(\frac{r_h^3 - r_o^3}{3} \right)$$

where ρ is the mass density of the disk material.

The ratio σ_o / σ_u of the imposed radial stress σ_o, and the hoop stress σ_u of Eq. 15 is then

$$\frac{\sigma_o}{\sigma_u} = \frac{F_B + F_R}{2 \pi r_o y_o} \quad \frac{1}{\rho \omega^2 r_o^2}$$

For the present example with $\rho_B = \rho$, $\varkappa = 0.5$, and $\zeta = 0.45$,

$$\frac{\sigma_o}{\sigma_u} = 0.719$$

The disk material is supposed to have a specific weight of 0.296 lb/in.^{3}, a coefficient of thermal expansion $\alpha = 8 \, (10^{-6}) \, \%/^{\circ}F$, and a modulus of elasticity $E = 25.5 \, (10^6)$ psi. For the chosen peripheral speeds of 800, 1000, and 1200 ft/sec, at the mean radius $r_{av} = 6.1$ in. of the blading, the reference hoop stresses σ_u are then 47,480; 74,200; and 106,900 psi, respectively. Further, for $\Delta T_o = 710^{\circ}F$, the reference thermal stress $\alpha \Delta T_o E$ equals 145,000 psi.

The centrifugal stresses at the different radii are obtained from Eqs. 16 with the dimensionless coefficients R_C and H_C of Figs. 3 and 4

for $y_o/y_i = 0.5$. The dimensionless coefficients R_T and H_T of Fig. 10 for $y_o/y_i = 0.5$ are used to determine the thermal stresses from Eqs. 22. The total stresses σ_r and σ_t in radial and tangetial directions, respectively, are

$$\sigma_r = (\sigma_r)_C + (\sigma_r)_T$$

$$\sigma_t = (\sigma_t)_C + (\sigma_t)_T$$

It is customary to apply the principle of maximum shear-strain energy for the establishing of the equivalent uniaxial tensile stress σ_e, which is equivalent to the biaxial stress condition in disks [11]. For the stresses σ_r and σ_t

$$\sigma_e = \sqrt{\sigma_r^2 + \sigma_t^2 - \sigma_r \sigma_t}$$

This equivalent stress is to be compared with the permissible stresses obtained from uniaxial tests.

Fig. 14 shows the equivalent stresses in the disk for the three chosen peripheral speeds. The curves labeled σ_e represent the equivalent stresses due to the combined effects of rotation and temperature gradients, whereas the curves marked $(\sigma_e)_C$ give the equivalent stresses because of centrifugal effects only.

DISCUSSION

The equivalent stresses $(\sigma_e)_C$ of Fig. 14 because of the rotation of the disk are nearly uniform along the radius and change with the square of the peripheral speeds of the disk. At the outer diameter of the disk the assumed temperature distribution produces compressive tangential stresses of 99,350 psi. Centrifugal effects produce tensile stresses at this station in tangential direction of 35,600, 55,600 and 80,000 psi for the peripheral speeds of 800, 1000, and 1200 ft/sec. respectively.

Thus the total equivalent stresses σ_e at the outer diameter of the disk are nearly equal for the three wheel speeds. In fact, the total equivalent stress at 1000 ft/sec is by 2000 psi lower than that at 800 ft/sec, and an increase of the wheel speed from 1000 to 1200 ft/sec increases σ_e only by about 4000 psi.

However, the total equivalent stress increases rapidly toward the outer radius of the disk, especially at the lower wheel speeds. Similar to the conditions that occur with local stress concentration the peak stresses at the outer radius will be reduced by plastic deformations since most disk materials have high ductility. It is doubtful therefore whether the peak stresses at the outer radius should be used as the criterion for the safety of the disk. Plastic deformations are likely to set up permanent stresses which lie between the values of σ_e and $(\sigma_e)_C$, or the total equivalent stresses will be about 60,000, 70,000, and 83,000 psi for the wheel speeds of 800, 1000, and 1200 ft/sec. In addition to these stresses the maximum values of σ_e at radius ratios between 0.3 and 0.4 must be compared with the permissible stresses at the prevailing disk temperatures.

CONCLUSIONS

The centrifugal and thermal stresses in conical turbine disks can be established with ease with the presented method. The assumed temperature distributions appear to be realistic, and the other assumptions of the calculating procedure are not at variance with common practice.

Uncertainties exist, however, as to how the results of the calculations have to be used for the evaluating of the safety of a turbine disk. The new Propulsion Laboratory of the Naval Postgraduate School will have a Hot Spin Test Unit in which radial temperature distributions can be imposed on rotating disks. Since the test unit is equipped with the necessary instrumentation, it is hoped that tests in the near future will show more clearly how the theoretical results have to be interpreted by the design engineer.

APPENDIX I

MATHEMATICAL ANALYSIS OF DISK STRESSES

<u>Formulation</u>. The equilibrium of the forces acting on the element ABCD of the rotating disk of Fig. 2 is given by [6]

$$\frac{d}{dr}(r\, y\, \sigma_r) - y\, \sigma_t + \rho\, \omega^2\, r^2\, y = 0 \tag{1}$$

The stresses σ_r and σ_t, and the thermal expansion displace the element to A' B' C' D'. The radial displacement of the element at the radius r is denoted by u. The radial and tangential strains ϵ_r and ϵ_t of the element are then given by $\epsilon_r = du/dr$, and $\epsilon_t = u/r$, respectively [6]. Then

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\,\frac{u}{r} - (1+\nu)\,\alpha\, T\right] \\ \sigma_t &= \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\,\frac{du}{dr} - (1+\nu)\,\alpha\, T\right] \end{aligned} \tag{2}$$

where α represents the thermal coefficient of expansion of the disk material, and T the temperature at the radius r.

With the radius R of the apex of the conical disk of Fig. 2, and introducing the variable $t = r/R$, there is $r = Rt$ and

$$y = y_m\,(1-t) \tag{3}$$

Then, by introducing Eqs. 2 into Eq. 1, and assuming that α and E remain constant along the radius

$$\frac{d^2u}{dt^2} + \left(\frac{1}{t} - \frac{1}{1-t}\right)\frac{du}{dt} - \left(\frac{1}{t} + \nu\,\frac{1}{1-t}\right)\frac{u}{t}$$

$$= -\,\frac{1-\nu^2}{E}\,\rho\,\omega^2\,R^3 t - (1+\nu)\,\alpha\left(\frac{T}{1-t} - \frac{dT}{dt}\right) \tag{4}$$

This equation must be solved to obtain $u = u\,(r)$. The disk stresses are then known from Eqs. 2.

Solution. Eq. 4 has the form

$$F(u) = f_C(t) + f_T(t) \tag{5}$$

where F(u) represents the left-hand side of Eq. 4, and $f_C(t)$, $f_T(t)$ the first and second term, respectively, of its right-hand side. As the subscripts C and T indicate, the function $f_C(t)$ is due to the rotation ω of the disk, and $f_T(t)$ is due to the change of the temperature T in radial direction. Since Eq. 5 is linear, its general solution u can be expressed by

$$u = u_C + u_T \tag{6}$$

where u_C and u_T are solutions of

$$\begin{aligned} F(u_C) &= f_C(t) \\ F(u_T) &= f_T(t) \end{aligned} \tag{7}$$

Because Eqs. 2 are linear also in u and du/dt,

$$\begin{aligned} \sigma_r &= (\sigma_r)_C + (\sigma_r)_T \\ \sigma_t &= (\sigma_t)_C + (\sigma_t)_C \end{aligned} \tag{8}$$

The centrifugal stresses $(\sigma_r)_C$ and $(\sigma_t)_C$ are obtained by introducing the solution u_C of Eq. 6 into Eqs. 2, and the thermal stresses $(\sigma_r)_T$ and $(\sigma_t)_T$ are given by Eqs. 2 with the solution u_T of Eq. 7. Hence, the total stresses σ_r and σ_T can be determined by adding the centrifugal stresses of the disk at uniform temperature to the termal stresses of the non-rotating disk that has the temperature distribution T = T(r).

Eqs. 6 and 7 are linear differential equations of the form

$$F(u) = f(t) \tag{9}$$

Let u* be the solution of the abridged equation

$$F(u) = 0 \tag{10}$$

Then, if u** is a particular, non-trivial, integral of Eq. 9, the general solution of Eq. 9 is

$$u = u^{**} + K u^{*} \tag{11}$$

where K is a constant that depends on the boundary conditions.

Eq. 10 is a hypergeometric differential equation which can be solved by infinite series. Honegger [1] gives numerical solutions for Poisson's ratio $\nu = 0.3$, which have the form

$$Ku^{*} = k_1 \eta_1 + k_2 \eta_2 \tag{12}$$

The problem of determining the centrifugal and thermal stresses in conical disks is therefore reduced to finding particular solutions of Eqs. 6 and 7.

Formulas for Centrifugal Stresses. Honegger [1] has established a particular solution u** of Eq. 6. Combining this solution with Eq. 12 gives the following relations for the centrifugal stresses in rotating conical disks at uniform temperature:

$$\begin{aligned} (\sigma_r)_C &= \rho\, \omega^2 R^2 q_1 + A_1 \; q_2 + A_2 \; q_3 \\ (\sigma_t)_C &= \rho\, \omega^2 R^2 s_1 + A_1 \; s_2 + A_2 \; s_3 \end{aligned} \tag{13}$$

The quantities A_1 and A_2 are constants, and the dimensionless values of q and s are listed in Table I for $\nu = 0.3$, as functions of $t = r/R$.

For solid conical disks with $r_i = 0$ the constant A_2 must be zero. If the disk is subjected to a uniform radial stress σ_o at its outer radius r_o,

$$A_1 = \frac{\sigma_o - \rho\, \omega^2 R^2 q_1 (t_o)}{q_2 (t_o)} \tag{14}$$

where $q_1 (t_o)$ and $q_2 (t_o)$ are the values of q_1 and q_2 for $t = t_o = r_o/R$.

By relating the stresses to a characteristic reference stress it is possible to establish non-dimensional stress coefficients which are equal for geometrically similar disks, irrespective of their mass densities ρ and their speeds of rotation ω. The chosen reference stress is the hoop stress σ_r in a ring with small radial thickness, having the radius r_o and the mass density ρ of the disk, that rotates with ω. Since the ring is not subjected to radial stresses, from Eq. 1,

$$\sigma_u = \sigma_t = \rho\,\omega^2 r_o^2 = \rho\,\omega^2 R^2 t_o^2 = \rho\,V_o^2 \tag{15}$$

where $t_o = r_o/R$, and V_o is the peripheral velocity of the disk at the outer radius r_o. Then, from Eqs. 13, with Eqs. 14 and 15.

$$\begin{aligned}\frac{(\sigma_r)_C}{\sigma_u} &= R_{C_1} + \left(\frac{\sigma_o}{\sigma_u}\right) R_{C_2}\\ &= \frac{1}{t_o^2}\left(q_1 - \frac{q_1\,(t_o)}{q_2\,(t_o)}\,q_2\right) + \left(\frac{\sigma_o}{\sigma_u}\right)\frac{q_2}{q_2\,(t_o)}\end{aligned}$$

$$\begin{aligned}\frac{(\sigma_t)_C}{\sigma_u} &= H_{C_1} + \left(\frac{\sigma_o}{\sigma_u}\right) H_{C_2}\\ &= \frac{1}{t_o^2}\left(s_1 + \frac{q_1\,(t_o)}{q_2\,(t_o)}\,s_2\right) + \left(\frac{\sigma_o}{\sigma_u}\right)\frac{s_2}{q_2\,(t_o)}\end{aligned} \tag{16}$$

For chosen values of t_o the dimensionless coefficients R_C and H_C can be calculated with the values of q and s from Table I, for values of $t = r/R$ between zero and t_o. However, for $r_i = 0$, and $y_m = y_i$, from Eq. 3,

$$y_o = y_i\,(1 - t_o) \quad \text{and} \quad t_o = 1 - (y_o/y_i)$$

Hence, by choosing the thickness ratios y_o/y_i there are known the values of t_o, and because $r/r_o = t/t_o$, it is possible to determine the coefficients R_C and H_C as functions of r/r_o, with y_o/y_i being the sole parameter. Figs. 3 and 4 show these coefficients for Poisson's ratio $\nu = 0.3$. For disks with constant thickness, or $y_o/y_i = 1$, Eqs. 16 gives indeterminate results. However, from direct integration of Eq. 1 for y = constant

$$R_{C_1} = \frac{3+\nu}{8}\left(1-\left(\frac{r}{r_o}\right)^2\right)$$

$$H_{C_1} = \frac{3+\nu}{8} - \frac{1+3\nu}{8}\left(\frac{r}{r_o}\right)^2$$

$$R_{C_2} = H_{C_2} = 1$$

<u>Formulas for Thermal Stresses</u>. The temperature distribution in the conical disk of Fig. 6 is supposed to have the form

$$T = T_m + \Delta T_o \ (r/r_o)^n \tag{17}$$

If T_i and T_o are known at r_i and r_o,

$$\Delta T_o = \frac{T_o - T_i}{1-(r_i/r_o)^n} \quad \text{and} \quad T_m = T_i - \Delta T_o (r_i/r_o)^n$$

Eq. 17, rewritten,

$$T = T_m + \Delta T \ t^n \tag{18}$$

where, with

$$t_o = r_o/R,$$

$$\Delta T = \frac{\Delta T_o}{t_o^{\ n}} \tag{19}$$

For exponents n of Eq. 18 that are whole numbers, a particular integral u_T** of Eq. 7 is obtained with the series

$$u^{**}_T = (1 + \nu)\, \alpha\, R\, \Delta T \sum_{i=1}^{n+1} a_i\, t^i \tag{20}$$

The coefficients a_i are found by equating terms with equal powers of t in the expression

$$\sum_{i=1}^{n+1} \left\{ a_i t^i \left[1-\nu-i(i+1)\right] + a_i t^{i-1} \left[i^2-1\right] \right\} = nt^n-(n+1)t^{n+1}$$

Eq. 20 introduced in Eq. 11, and using Eq. 12, gives the following formulas for the thermal stresses in conical disks:

$$(\sigma_r)_T = \alpha\, \Delta T\, E \left[\sum_{i=1}^{n+1} \left(\frac{i + \nu}{1 - \nu}\, a_i t^{i-1} \right) - t^n \right]$$

$$+ B_1\, q_2 + B_2\, q_3 \tag{21}$$

$$(\sigma_t)_T = \alpha\, \Delta T\, E \left[\sum_{i=1}^{n+1} \left(\frac{1 + i\nu}{1-\nu}\, a_i\, t^{i-1} \right) - t^n \right]$$

$$+ B_1\, s_2 + B_2\, s_3$$

The values of q and s are given in Table I, and B_1, B_2 are constants. For conical disks without a central hole ($r_i = 0$) the constant B_2 is zero because $(\sigma_r)_T = (\sigma_t)_T$ at $r = 0$, and B_1 is obtained from the condition that $(\sigma_r)_T = 0$ at $r = r_o$, or $t = t_o$.

Dimensionless stress coefficients will be defined by relating the thermal stresses to the reference stress $\alpha\, \Delta T_o\, E$. This thermal stress occurs in a restrained bar of the disk material, which is heated up by the temperature ΔT_o. For solid disks the quantity ΔT_o is the tempera-

ture difference between the outer radius r_o and the center line.

With Eq. 19 and Eqs. 21, for solid conical disks,

$$\frac{(\sigma_r)_T}{\alpha \Delta T_o E} = R_T$$

$$= \sum_{i=1}^{n+1} \left[\frac{i+\nu}{1-\nu} a_i \frac{t^{i-1}}{t_o^n} \right] - \left(\frac{t}{t_o} \right)^n - \frac{q_2}{q_2(t_o)} \left\{ \sum_{i=1}^{n+1} \left[\frac{i+\nu}{1-\nu} a_i \frac{t_o^{i-1}}{t_o^n} \right] - 1 \right\}$$

$$\frac{(\sigma_t)_T}{\alpha \Delta T_o E} = H_T$$

$$= \sum_{i=1}^{n+1} \left[\frac{1+i\nu}{1-\nu} a_i \frac{t^{i-1}}{t_o^n} \right] - \left(\frac{t}{t_o} \right)^n - \frac{s_2}{q_2(t_o)} \left\{ \sum_{i=1}^{n+1} \left[\frac{1+i\nu}{1-\nu} a_i \frac{t_o^{i-1}}{t_o^n} \right] - 1 \right\}$$

For chosen values of $t_o = r_o/R$ the coefficients R_T and H_T are functions only of $t = r/R$ and n. However, since $t_o = 1-(y_o/y_i)$, and $t/t_o = r/r_o$, they can be represented with r/r_o as variable and y_o/y_i as parameter. Figs. 7 to 10 show R_T and H_T for values of n from unity to four, for Poisson's ratio $\nu = 0.3$. Indeterminate results are obtained from Eqs. 22 for $y_o/y_i = 1$. However, adapting the known expressions for a disk with constant thickness [7] to the temperature distribution of Eq. 17, gives

$$R_T = \frac{1 - (r/r_o)^n}{n + 2}$$

$$H_T = \frac{1 - (n+1)(r/r_o)^n}{n + 2} \tag{23}$$

APPENDIX II

TEMPERATURE DISTRIBUTION IN DISKS WITH CONSTANT THICKNESS

Fig. 12 shows a turbine disk with external cooling. Cold gas having the temperature T_c flows along one face of the disk. Hot gases at the temperature T_g pass through the turbine and transmit the heat q_o to the profile surfaces and the base of the blades per unit time. With an equivalent surface film coefficient h_o, referred to the outer rim surface of the disk.

$$q_o = 2 \pi r_o y h_o (T_g - T_o)$$

where T_o is the disk temperature at the outer radius r_o .

The quantity of heat dq, representing the difference of the heat flux through the disk at the radii r and r + dr of Fig. 12a, is transmitted to the coolant per unit time. With a film coefficient h_c on the face of the disk.

$$\frac{d}{dr}\left(2 \pi r y k \frac{dT}{dr}\right) dr = 2 \pi r \, dr \, h_c (T-T_c)$$

where T is the temperature of the disk at the radius r, and k the coefficient of thermal conductivity of the disk material. For an assumed constant temperature T_c of the coolant, with $T = T_c + \theta$, and introducing the variable x defined by

$$x = r \sqrt{\frac{h_c}{yk}} \tag{24}$$

there is

$$\frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - \theta = 0 \tag{25}$$

This modified Bessel equation of order zero has the solution [8]

$$\theta = C \, I_o(x)$$

where I_o is the modified Bessel function of zero order. The constant C is obtained from the condition that the heat q_o added to the disk at r_o must be conducted away by the disk, or

$$h_o \ (T_g - T_o) = k \left(\frac{dT}{dr} \right)_{r=r_o}$$

With $dT/dr = d\theta/dr$, and from Eqs. 24 and 26

$$\left(\frac{d\theta}{dr} \right)_{r=r_o} = \left(\frac{d\theta}{dx} \right)_{x=x_o} \sqrt{\frac{h_c}{y\,k}} = C \, \frac{dI_o(x_o)}{dx} \sqrt{\frac{h_c}{y\,k}} \qquad (27)$$

Since the derivative of the modified Bessel function of zero order equals the modified Bessel function I_1 (x) of the first order [8], with Eq. 26 and C from Eq. 27,

$$T = T_c \qquad \frac{Tg - T_c}{\frac{1}{h_o} \sqrt{\frac{h_c \, k}{y}} \, \frac{I_1(x_o)}{I_o(x)} + \frac{I_o(x_o)}{I_o(x)}} \qquad (28)$$

where x and x_o are given by Eq. 24 for r and r_o, respectively.

The functions $I_o(x)$ and $I_1(x)$ are tabulated in Ref. [9] for values of x from zero to 16. Both functions are non-periodic. For values of x larger than six both functions are large, but the ratio $I_1(x)/I_o(x)$ tends toward unity. Since in many cases the values of x_o are large, there is approximately

$$\frac{Tg - T_o}{Tg - T_c} \cong \frac{\frac{1}{h_o} \sqrt{\frac{h_c \, k}{y}}}{1 + \frac{1}{h_o} \sqrt{\frac{h_c \, k}{y}}}$$

This ratio should be large to operate a turbine at high gas temperatures T_g for specified disk temperatures T_o. For given conditions it is therefore advisable to use a material with a high coefficient of thermal conductivity.

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TABLE I

Values of Coefficients q and s for ν=0.3

(Adapted from Ref. 1)

t=r/R	q_1	q_2	q_3	s_1	s_2	s_3
0	0.1653	1.4286	$-\infty$	0.1653	1.4286	$+\infty$
0.05	0.1709	1.4857	-136.9559	0.1693	1.4861	147.2306
0.10	0.1750	1.5494	- 33.3406	0.1724	1.5110	38.8791
0.15	0.1777	1.6209	- 14.3626	0.1747	1.5582	18.3736
0.20	0.1790	1.7000	- 7.7802	0.1762	1.6099	10.9670
0.25	0.1787	1.7879	- 4.7692	0.1768	1.6670	7.4395
0.30	0.1770	1.8901	- 3.1604	0.1766	1.7308	5.4835
0.40	0.1693	2.1418	- 1.5824	0.1737	1.8824	3.4593
0.50	0.1556	2.4901	- 0.8747	0.1673	2.0813	2.4714
0.60	0.1362	3.0061	- 0.5017	0.1577	2.3560	1.9077
0.70	0.1109	3.8626	- 0.2842	0.1446	2.7835	1.5505
0.75	0.0961	4.5626	- 0.1989	0.1368	3.1055	1.4198
0.80	0.0798	5.5527	- 0.1490	0.1282	3.5439	1.3066
0.85	0.0620	6.8956	- 0.1002	0.1187	4.1384	1.2121
0.90	0.0428	10.5725	- 0.0603	0.1084	5.5164	1.1319
0.95	0.0221	20.5604	- 0.0274	0.0973	9.0549	1.0614
1.00	0	∞	0	0.0853	∞	1.0

CAPTIONS FOR ILLUSTRATIONS

Fig. 1 Typical Turbine Wheel.

Fig. 2 Stresses in Conical Disks.

σ_r = radial stress

σ_t = tangential stress

σ_o = impressed radial stress at r_o

u = radial deflection of element ABCD at radius r

Fig. 3 Dimensionless Stress Coefficients R_{C_1} and H_{C_1} of Eqs. 16 for Centrifugal stresses in Solid Conical Disks for Poisson's Ratio $\nu = 0.3$ (Influence of Disk Shape).

Fig. 4 Dimensionless Stress Coefficients R_{C_2} and H_{C_2} of Eqs. 16 for Centrifugal Stresses in Solid Conical Disks for Poisson's Ratio $\nu = 0.3$ (Influence of Impressed Radial Stress σ_o at outer Radius r_o)

Fig. 5 Assumed Radial Temperature Distribution $T = T_m + \Delta T\ (r/r_o)^n$ for different Values of n.

Fig. 6 Radial Temperature Distribution in Conical Disk.

Fig. 7 to 10 Dimensionless Stress Coefficients R_T and H_T of Eqs. 22 for Thermal Stresses in Solid Conical Disk for Poisson's Ratio $\nu = 0.3$.

Fig. 7 $n = 1$

Fig. 8 $n = 2$

Fig. 9 $n = 3$

Fig. 10 $n = 4$

Fig. 11 Dimensionless Stress Coefficients R_T and H_T of Eqs. 23 for Thermal Stresses in Disks with Constant Thickness for Poisson's Ratio $\nu = 0.3$.

Fig. 12 Cooling of Turbine Disk.

(a) Disk Dimensions and Cooling Scheme.

(b) Temperature Distribution.

Fig. 13 Temperature Distribtuion in Disk of Fig. 12 for $T_g = 1500^oF$, $T_c = 500^oF$.

(Points) : $T = T_i + \Delta T_{o1} (r/r_o)^4$

(Circles) : $T = T_i + \Delta T_{o2} (r/r_o)^{2.5}$

Fig. 14 Equivalent Stresses in Turbine Disk of Fig. 12 for $T = T_i + \Delta T_{o1}(r/r_o)^4$

σ_e = equivalent total stress

$(\sigma_e)_C$ = equivalent centrifugal stress

——— for U_{av} = 800 ft/sec

— — — for U_{av} = 1000 ft/sec

— - — for U_{av} = 1200 ft/sec

U_{av} = peripheral wheel speed at mean radius $r_{av} = r_h + h/2$ = 6.1 in. of blading of Fig. 1.

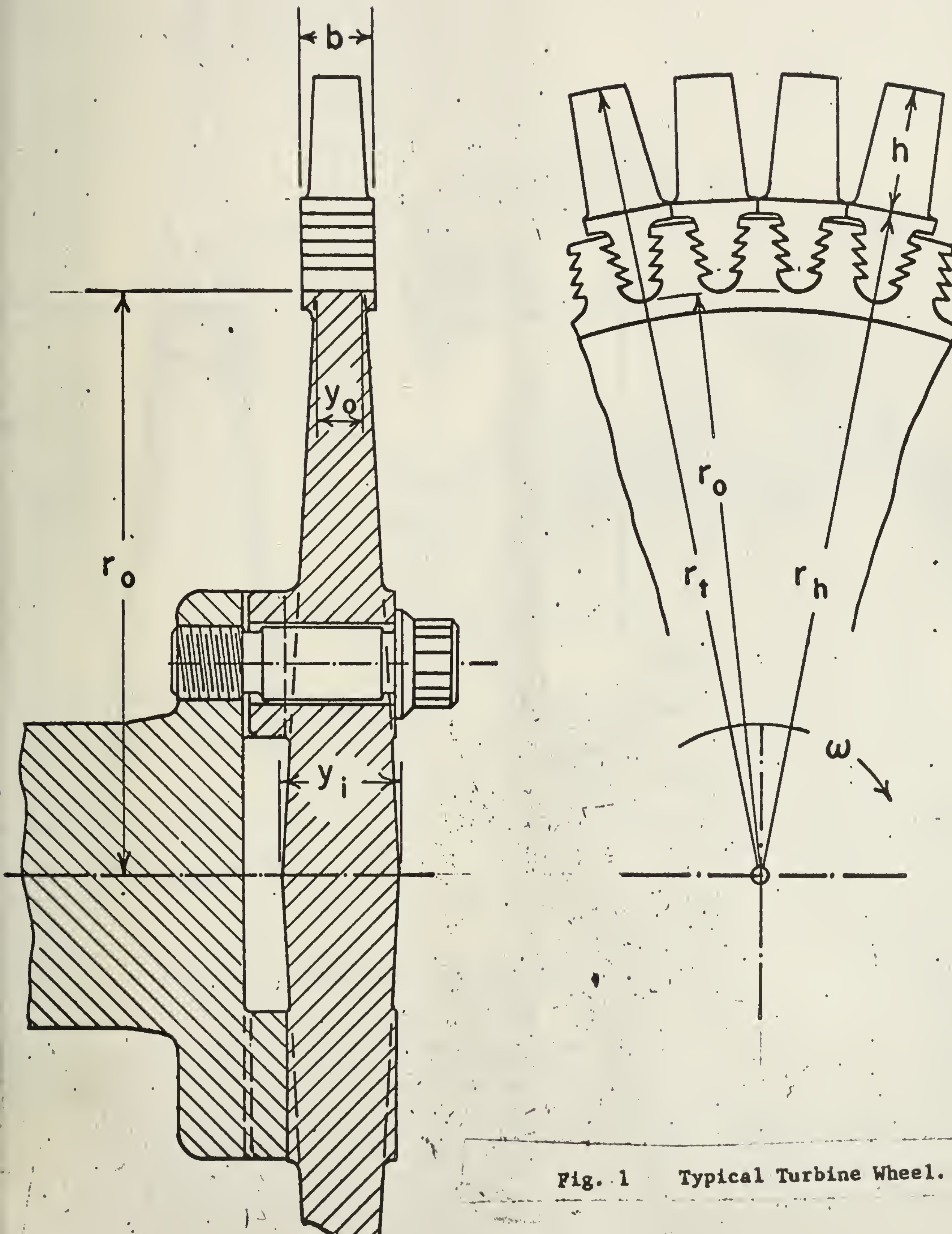

Fig. 1 Typical Turbine Wheel.

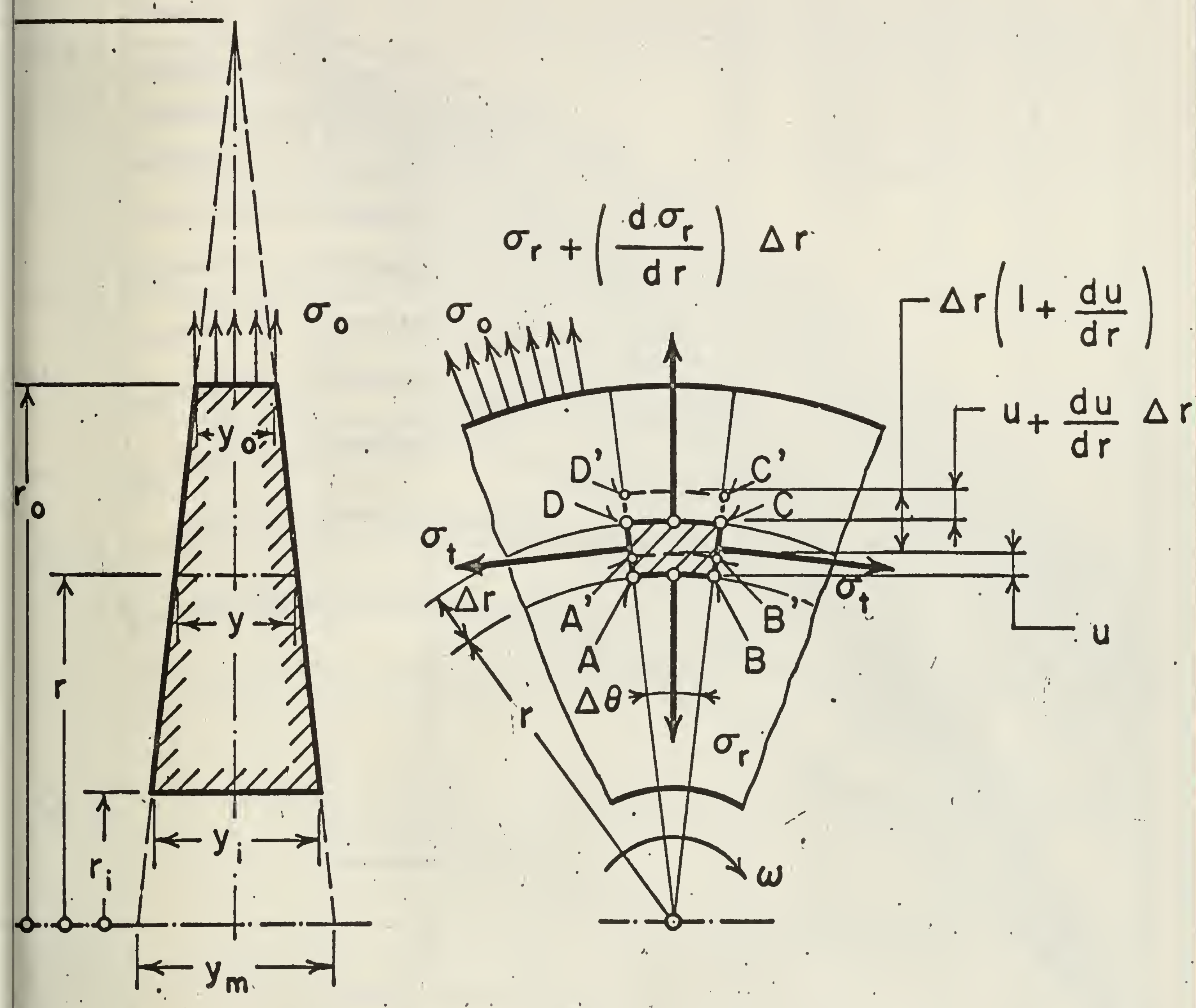

Fig. 2 Stresses in Conical Disks.

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σ_t = tangential stress

σ_o = impressed radial stress at r_o

u = radial deflection of element ABCD at radius r

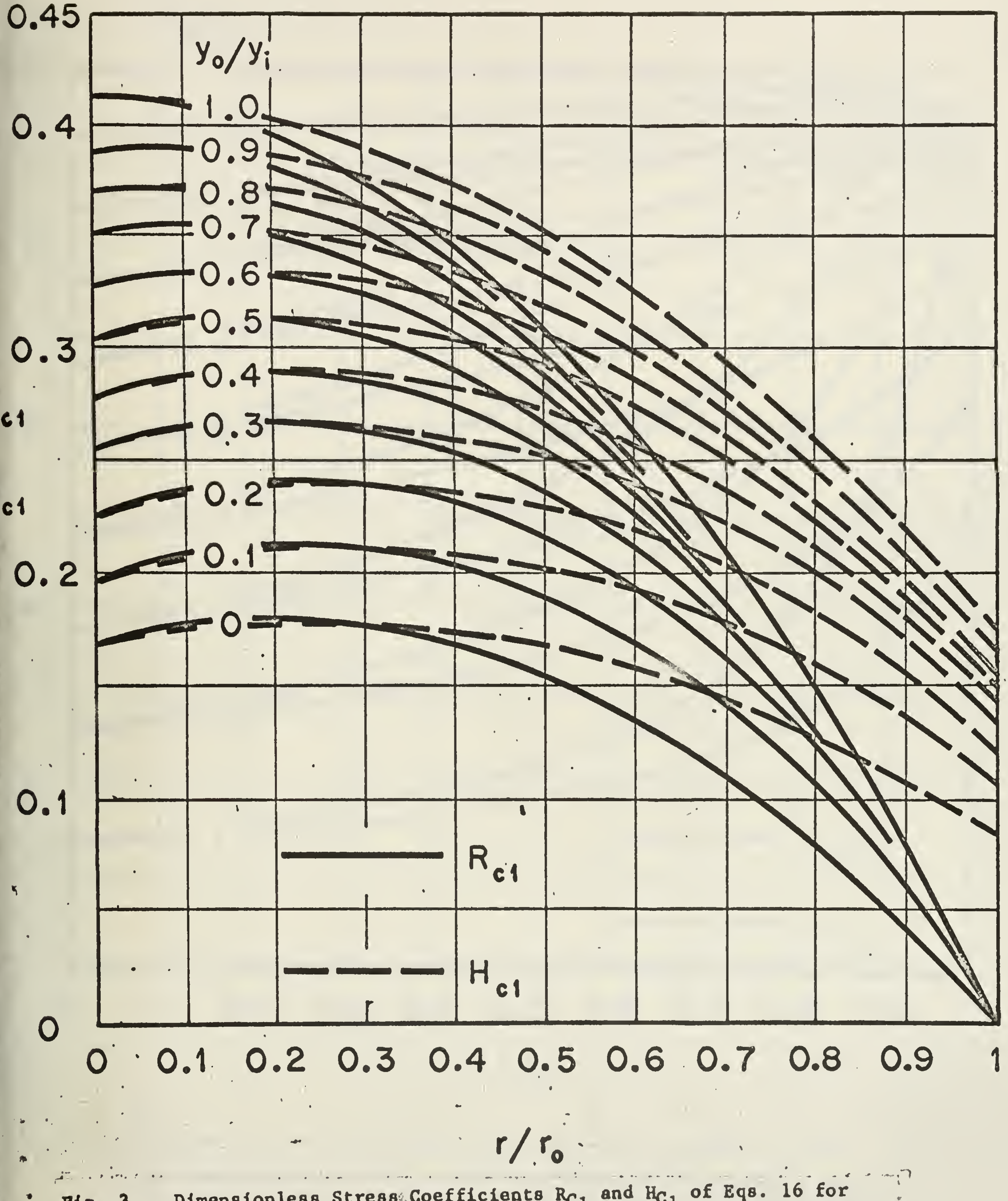


Fig. 3 . Dimensionless Stress Coefficients R_{C_1} and H_{C_1} of Eqs. 16 for Centrifugal stresses in Solid Conical Disks for Poisson's Ratio $\nu = 0.3$ (Influence of Disk Shape).

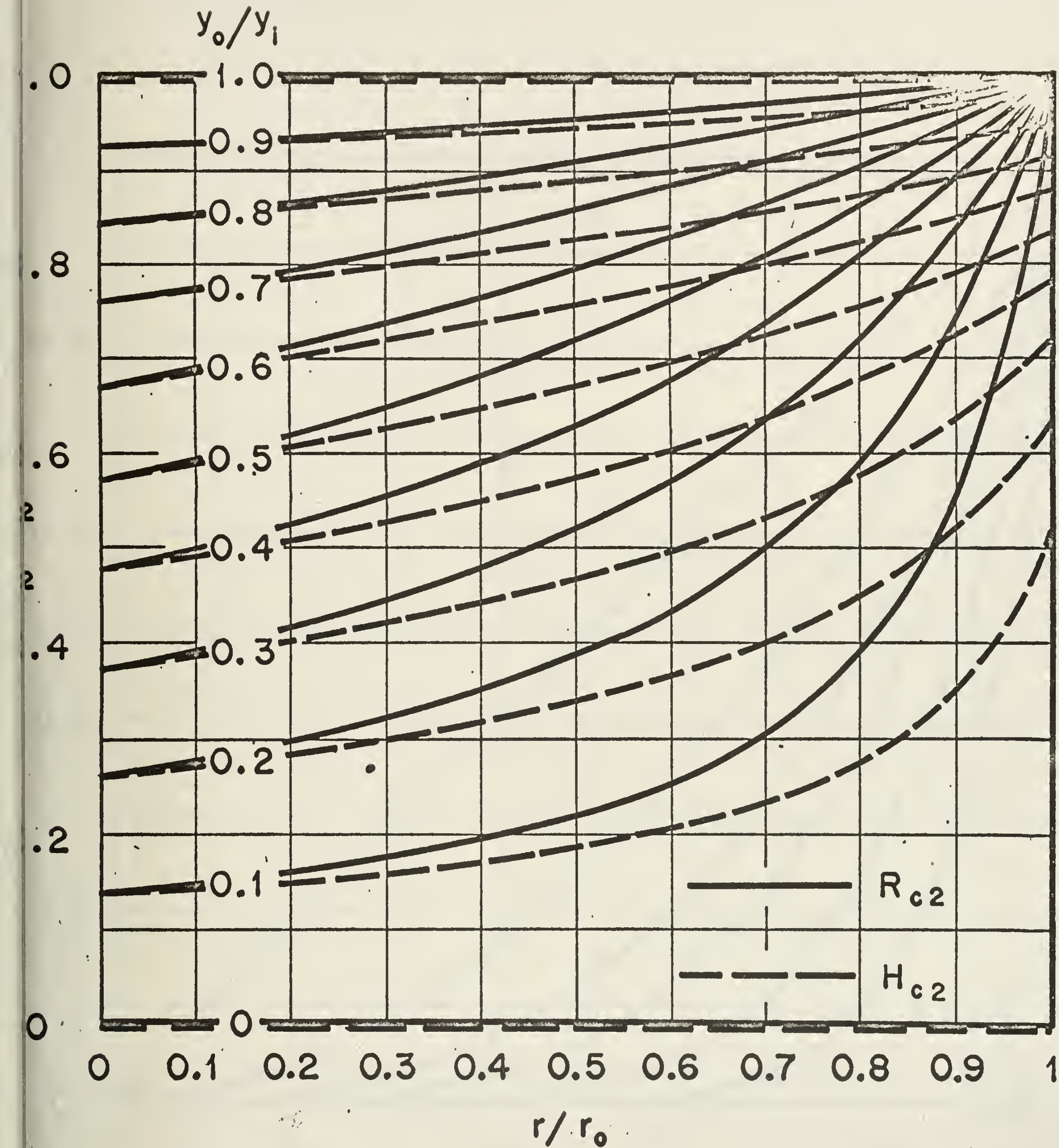


Fig. 4 Dimensionless Stress Coefficients R_{C2} and H_{C2} of Eqs. 16 for Centrifugal Stresses in Solid Conical Disks for Poisson's Ratio $\nu = 0.3$ (Influence of Impressed Radial Stress σ_o at outer Radius r_o).

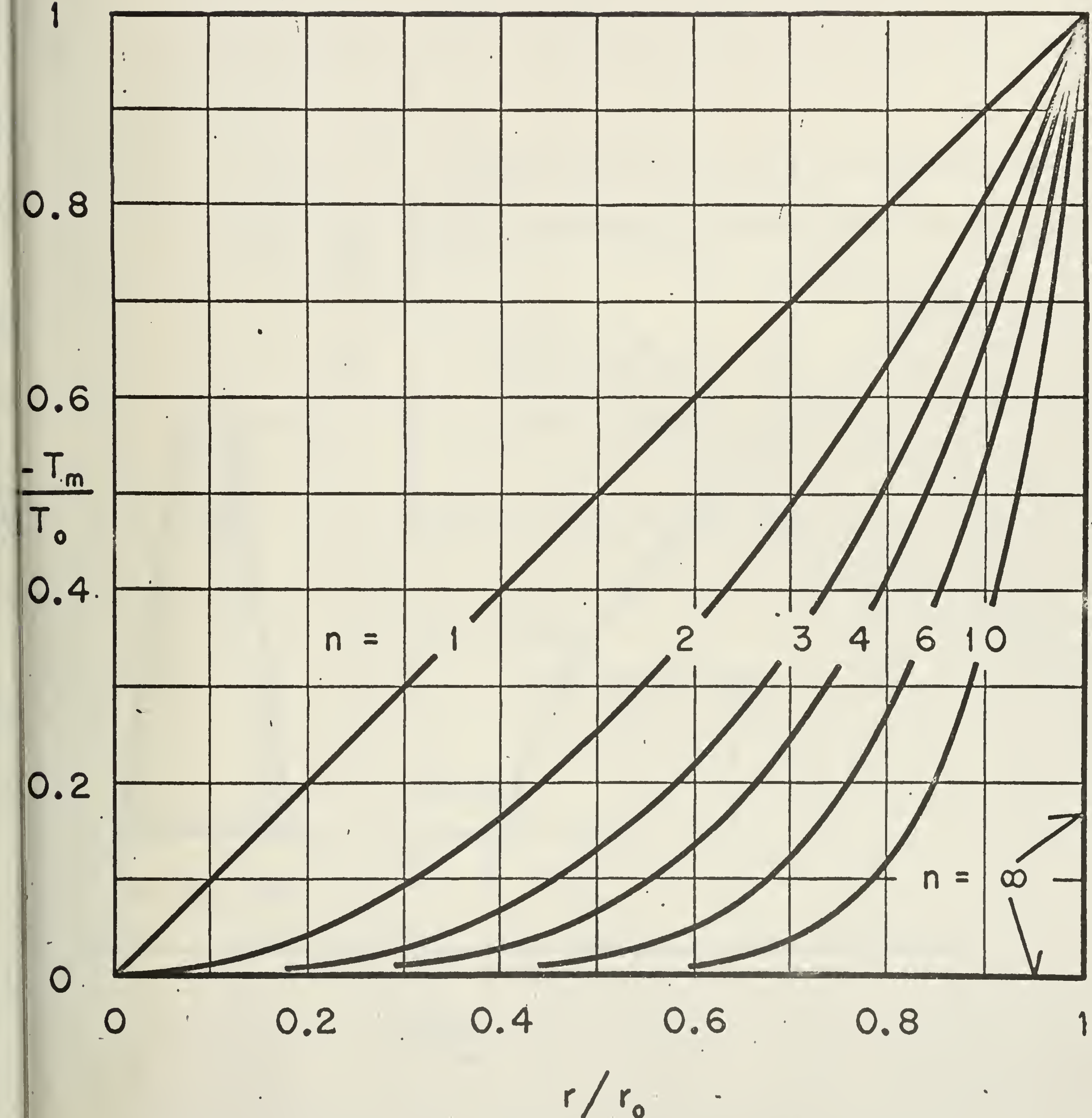


Fig. 5 Assumed Radial Temperature Distribution $T = T_m + \Delta T\ (r/r_o)^n$ for different Values of n.

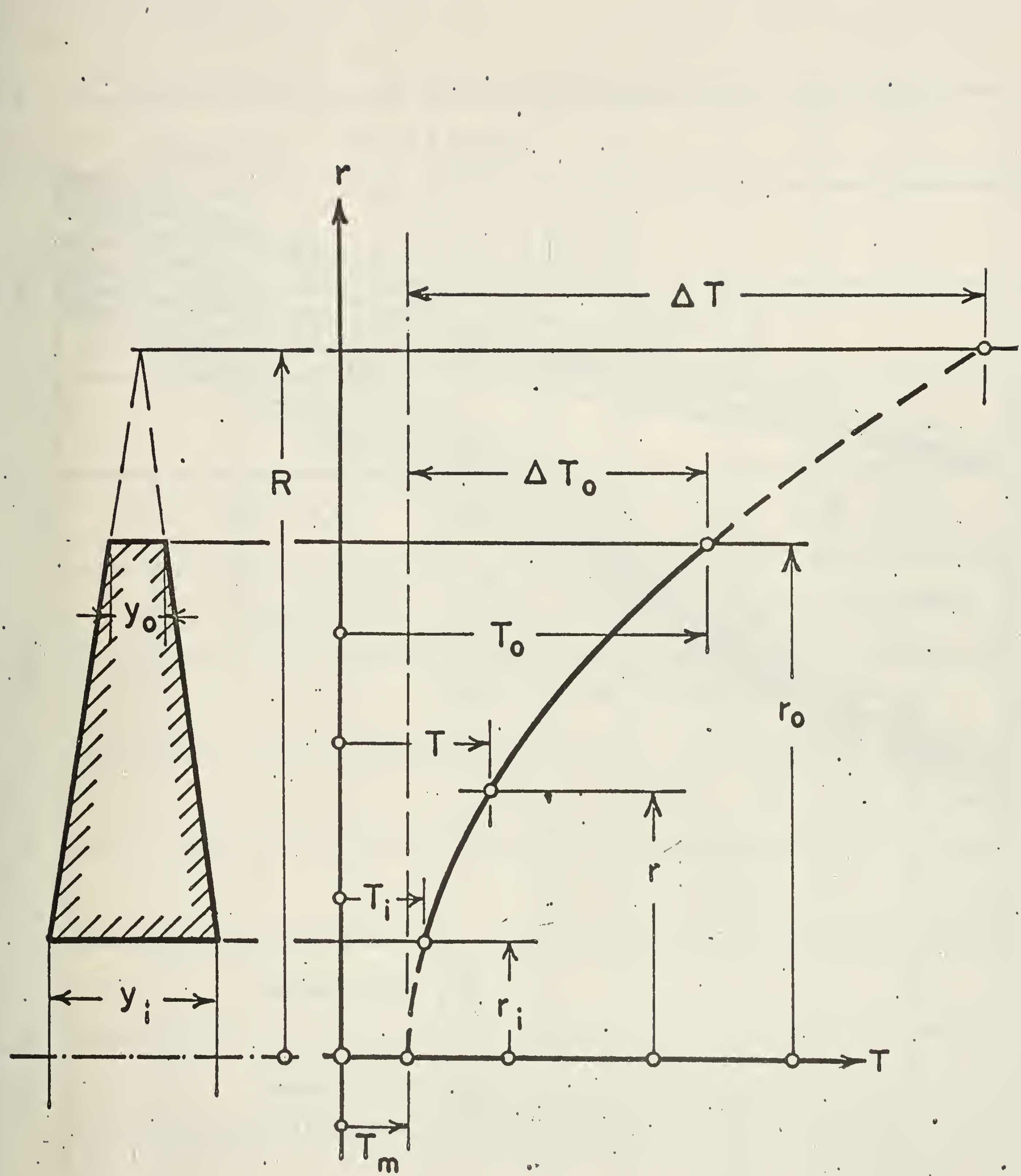


Fig. 6 Radial Temperature Distribution in Conical Disk.

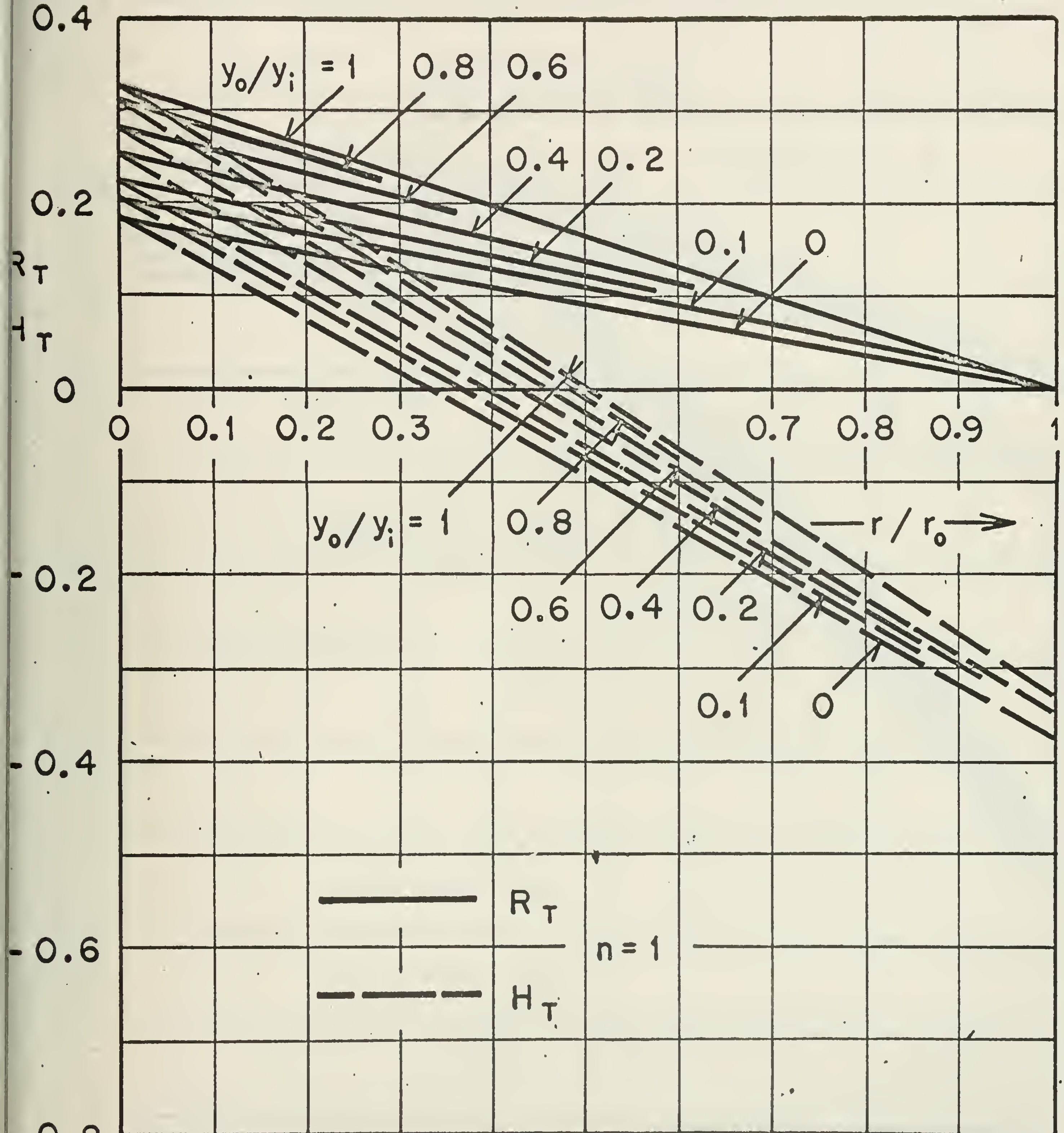


Fig. 7 to 10 Dimensionless Stress Coefficients R_T and H_T of Eqs. 22 for Thermal Stresses in Solid Conical Disk for Poisson's Ratio ν = 0.3.

Fig. 7 n = 1

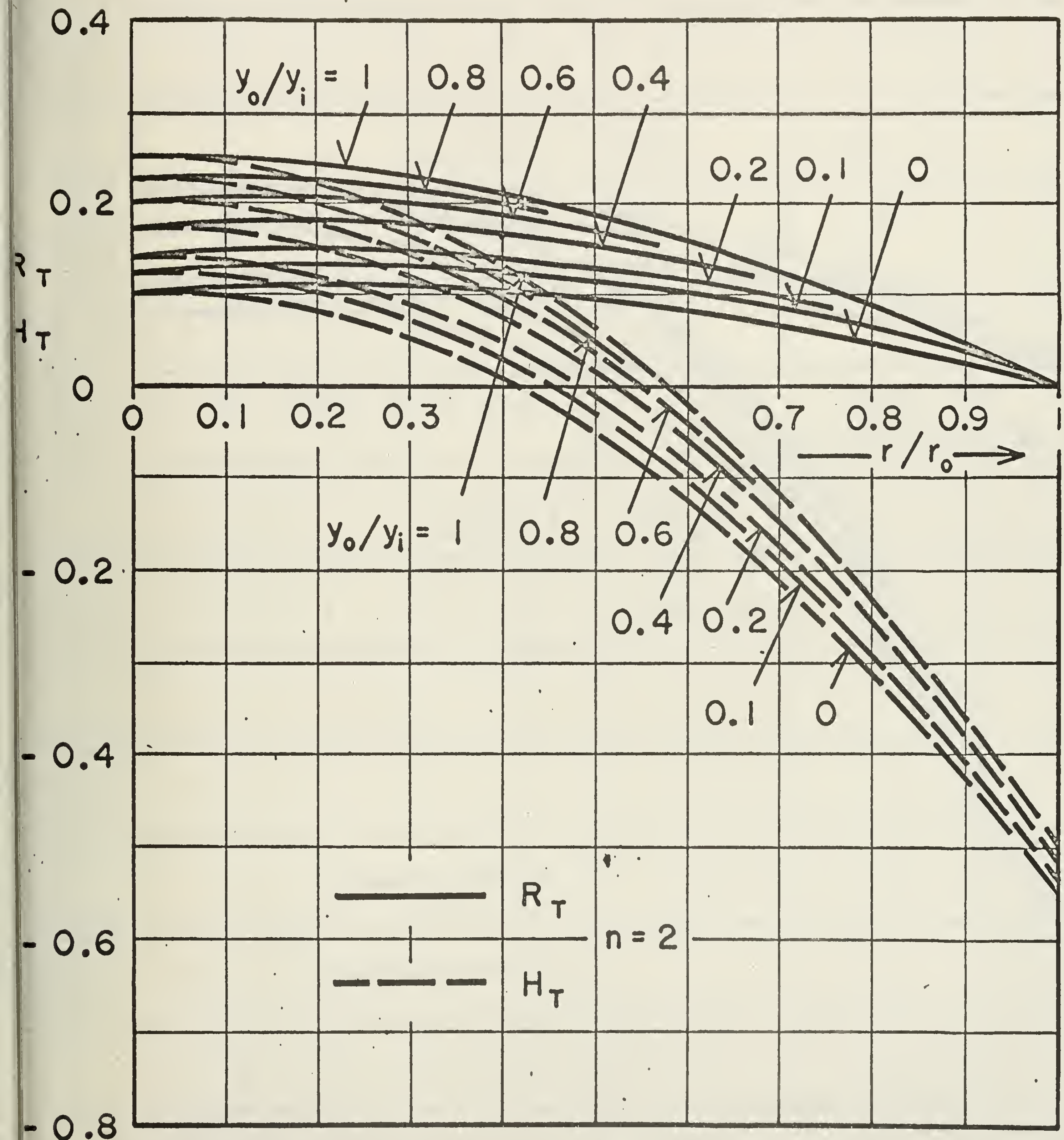
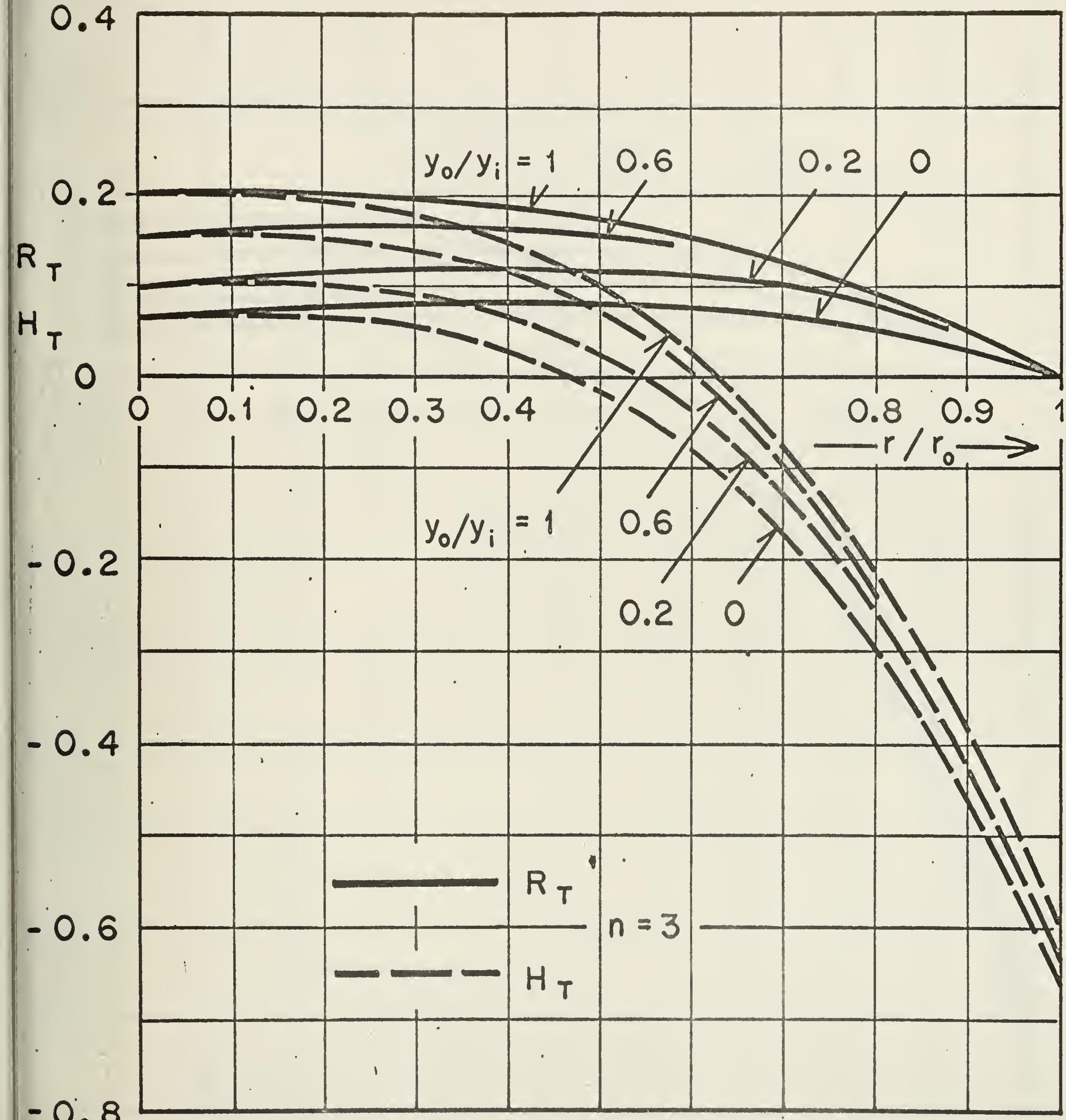

0.4
0.2
0
- 0.2
- 0.4
- 0.6
- 0.8
R_T
H_T
0
0.1
0.2
0.3
0.7
0.8
0.9
1
r/r_o
$y_o/y_i = 1$
0.8
0.6
0.4
0.2
0.1
0
$y_o/y_i = 1$
0.8
0.6
0.4
0.2
0.1
0
R_T
H_T
n = 2

Fig. 8 $n = 2$



0.4
0.2
R_T
H_T
0
-0.2
-0.4
-0.6
-0.8
y_o/y_i = 1
0.6
0.2
0
0
0.1
0.2
0.3
0.4
0.8
0.9
1
r/r_o
y_o/y_i = 1
0.6
0.2
0
R_T
n = 3
H_T

Fig. 9 n = 3

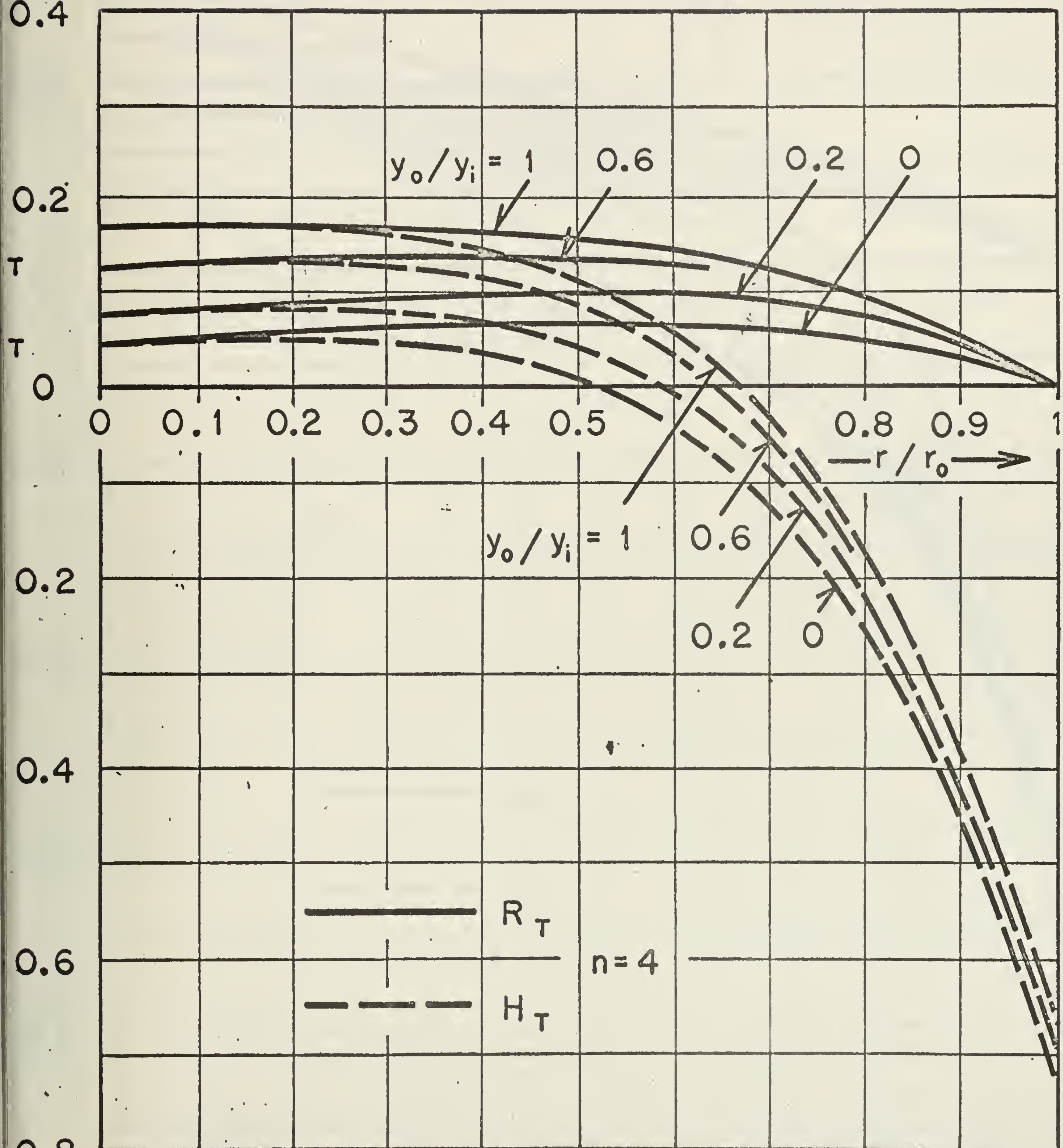
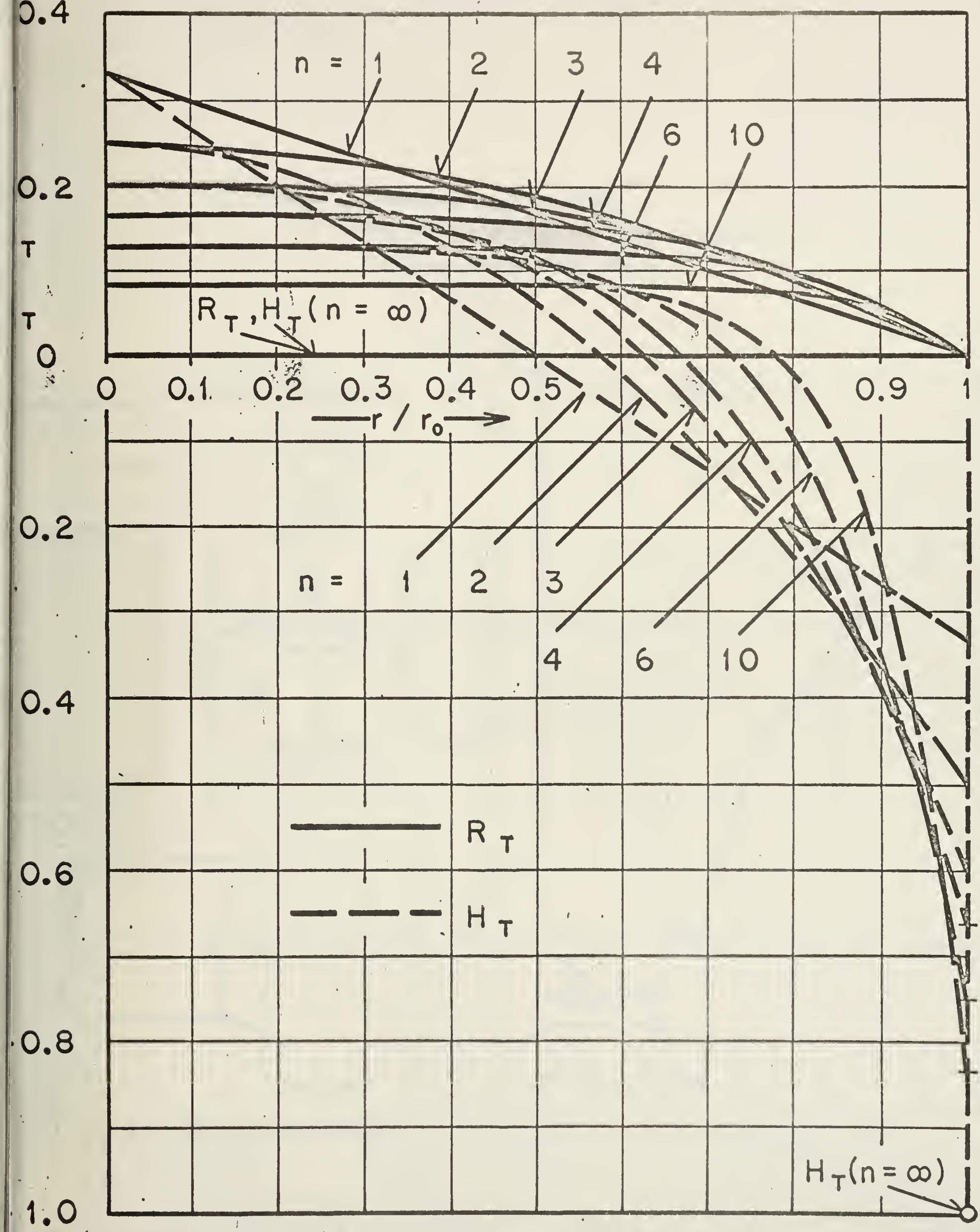

0.4
0.2
T
T
0
0.2
0.4
0.6
0.8
0 0.1 0.2 0.3 0.4 0.5 0.8 0.9 1
r / r₀
y₀ / yᵢ = 1 0.6 0.2 0
y₀ / yᵢ = 1 0.6
0.2 0
R T
H T
n = 4

Fig. 10. $n = 4$



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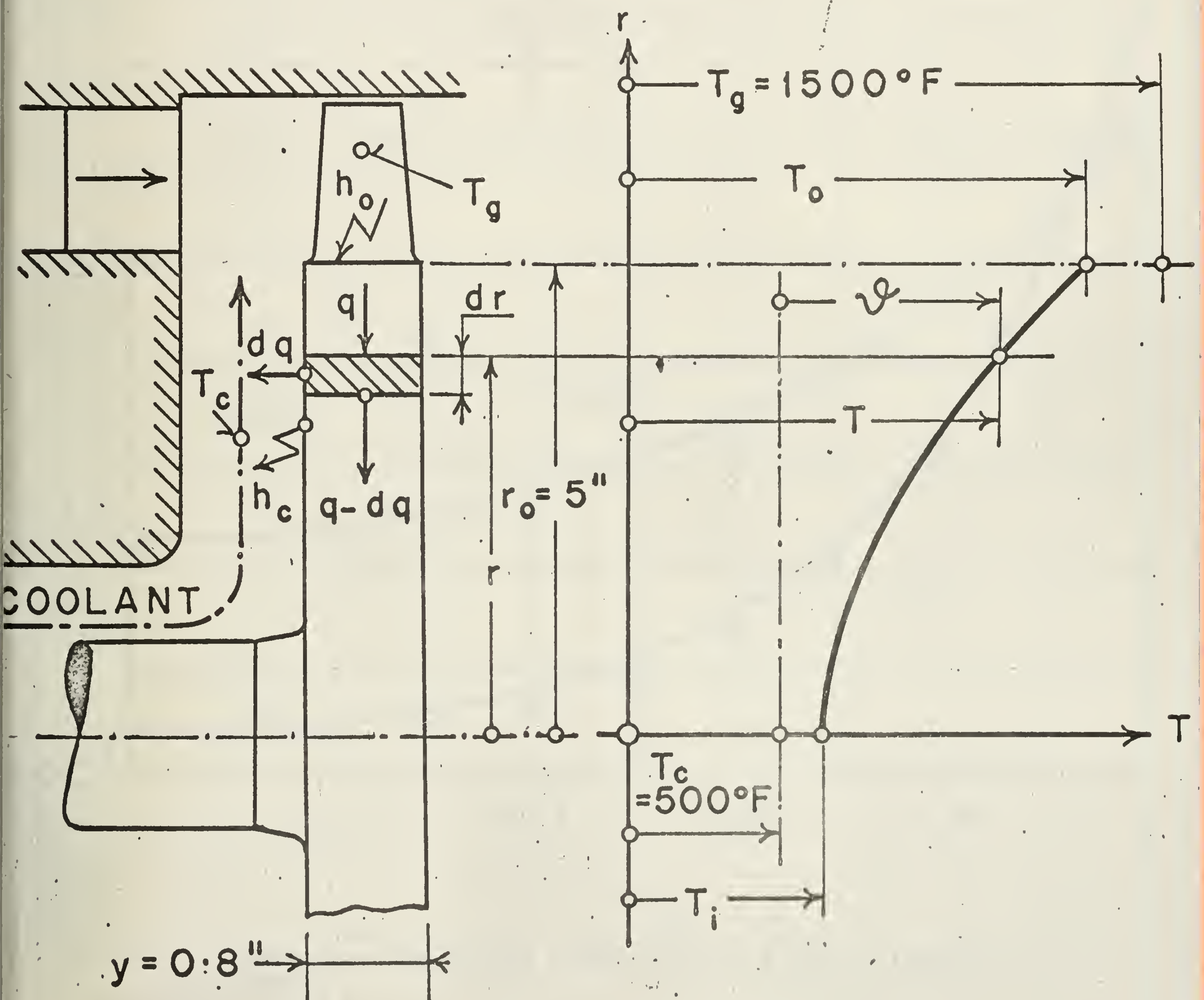

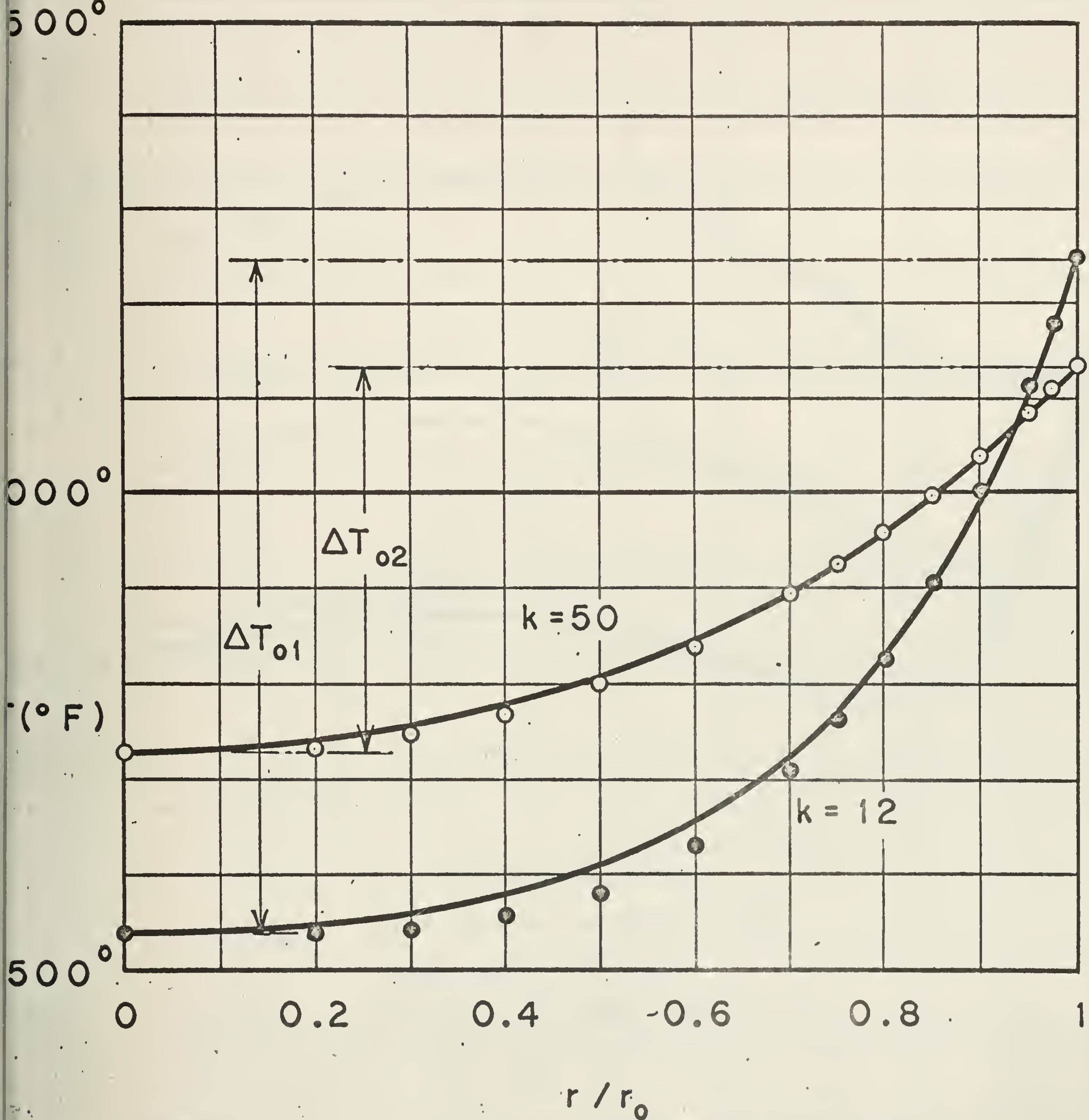

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(Circles): $T = T_i + \Delta T_{o2} (r/r_o)^{2.5}$

U_{av} = peripheral wheel speed at mean radius $r_{av} = r_h + h/2 = 6.1$ in. of blading of Fig. 1.

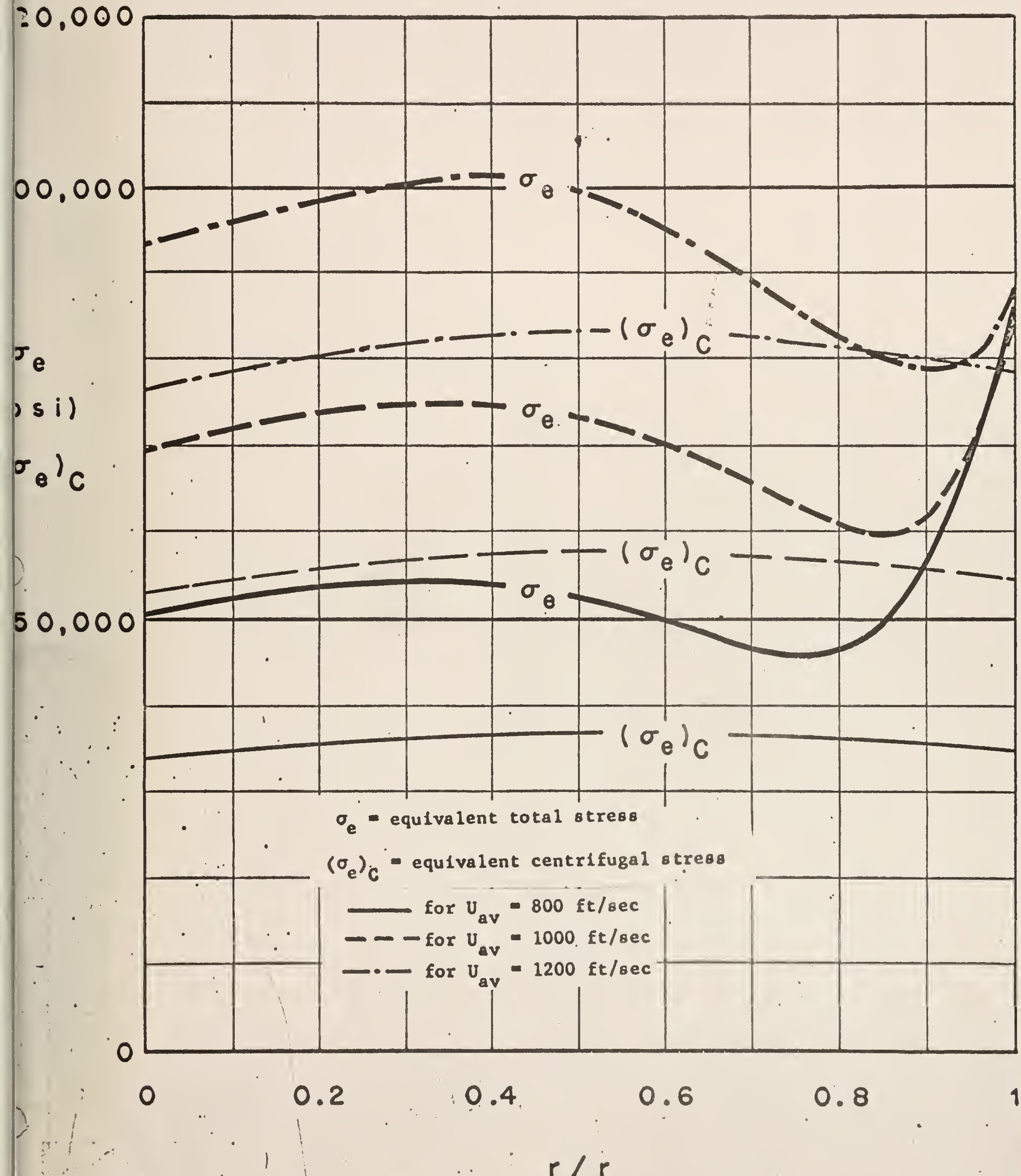

Fig. 14 Equivalent Stresses in Turbine Disk of Fig. 12 for $T = T_i + \Delta T_{oi}(r/r_o)^4$